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Quantizing Noise of ΔM/PCM Encoders

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We consider the pulse-code-modulation encoder that contains a delta modulator for analog-to-digital conversion, and a finite impulse response digital filter that suppresses high-frequency components of the delta modulation signal. A PCM word generator produces fixed-length binary code words by rounding and amplitude limiting the filter output samples. The quantizing noise of the resulting PCM signal has four components: delta modulation slope overload noise, filtered delta modulation granular noise, amplitude overload noise, and word generator roundoff noise. We analyze the total quantizing noise for the case where the encoder input is a Gaussian random process and the digital filter impulse response is uniform (all coefficients equal). Such filters possess important implementation advantages and appear to be near optimal with respect to signal-to-noise performance. Our analysis results in curves which show the relationship of signal-to-noise ratio to filter order, delta modulation sampling rate, and PCM word length.

I. INTRODUCTION

A new approach to digital encoding of continuous waveforms employs digital hardware to unite the economy of single-integration delta modulation (ΔM) with the efficiency of pulse code modulation (PCM). A finite impulse response digital filter suppresses the granular noise component of the ΔM representation of a continuous signal, and a word generator truncates the binary coded filter output to produce PCM code words of desired length. This encoding method controls the precision of the digital code by means of the ΔM speed and the filter order rather than with the resolution of the multibit quantizer that appears in conventional PCM encoders. This is a desirable substitution in view of current technology in which the cost of high-speed digital circuitry is rapidly declining.

This method of ΔM/PCM encoding, which was originally proposed by Goodman,[1] has been applied to speech encoding by Freeny, et al.,[2,3]

and to video by Kaneko and Ishiguro.[4] Previous theoretical results[1] focus on the filtering of the ΔM granular noise, but provide little insight into the important influence of ΔM slope overload and PCM amplitude overload on encoder design. Assuming the encoder input is a sample function of a Gaussian random process, the present paper analyzes the effects of the overload components of the quantizing distortion. It demonstrates that amplitude overload noise can be reduced if least significant bits of the filter output are truncated.

We focus our attention on "uniform filter encoders," in which all filter impulse response coefficients are unity. Such encoders offer significant practical advantages, and they appear to be near optimal with respect to signal-to-noise performance. For such encoders, we show how performance varies with filter order, ΔM speed, and PCM word length, and we demonstrate the application of our results to the design of practical encoders.

II. SIGNAL PROCESSING OPERATIONS

The block diagram of Fig. 1 shows the operations involved in transforming the continuous signal $y(t)$ to a uniformly quantized M-bit PCM sequence. Digital logic may be added to convert this sequence to a nonuniform PCM format.[5] The single-integration delta modulator of Fig. 2 converts $y(t)$ to a sequence of pulses with amplitude $+1$ or -1 at the rate $f_s = 1/\tau$ per second. The feedback loop is an ideal integrator with gain factor δ, while the up/down counter obtains a digital replica of $x(t)$, the ΔM approximation signal. The output of the Nth-

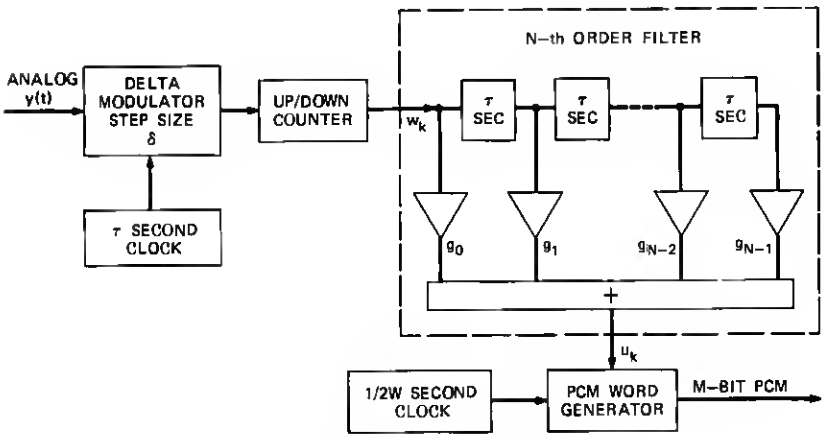

Fig. 1—Encoder block diagram.

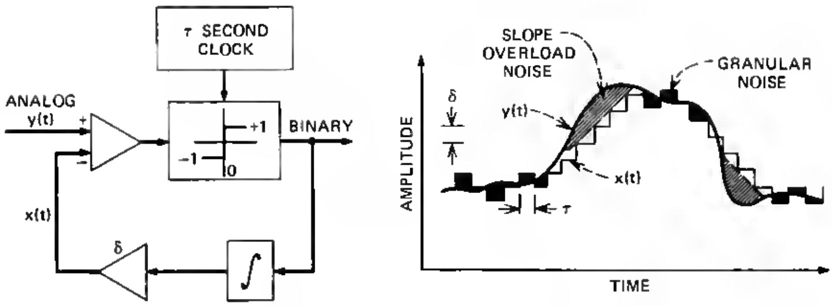

Fig. 2—Delta modulator.

order digital filter is a weighted sum of consecutive ΔM approximations to $y(t)$.

Although the filter outputs range over a discrete set, the number of possible filter outputs is unlimited because there is no fundamental restriction on the amplitude range represented by a delta modulator. It follows that, with the PCM word length prespecified, an additional quantizing operation is required. This quantization is performed by the word generator which restricts to 2^M the number of possible coder output words. The word generator introduces amplitude overload and it may also add to the granular quantization error by rounding off least significant bits of the filter output.

The filter and word generator are controlled by a clock which causes coder output words to be generated at the rate $2W$ per second, where W is the bandwidth of the analog input. Hence, the data rate of the coder is $2MW$ bits/second and the PCM sequence may be decoded as if it were produced by a conventional encoder consisting of a $1/2W$-second sampler and a uniform quantizer with 2^M output levels.

III. THE PCM QUANTIZATION LEVELS

With the filter coefficients, g_i, integers as in a practical realization, the filter output at $t = k\tau$ is the integer

$$u_k = \sum_{i=0}^{N-1} g_i w_{k-i} \tag{1}$$

where $\{w_j\}$ is the sequence of counter outputs. Because $w_j = w_{j-1} \pm 1$, the parity of the filter input alternates between even and odd at each ΔM sampling instant. It follows that if $f_s/2W$, the ratio of ΔM sampling rate to PCM sampling rate, is an even integer, the parities of

$w_k, w_{k-1}, \cdots, w_{k-N+1}$ are invariant at the PCM sampling instants. Hence, the parity of u_k is the same at all PCM sampling instants. Because odd-parity filter outputs lead to an easily implemented word generator, we restrict our attention to encoders in which w_k and u_k are both odd integers at the PCM sampling instants. The filter coefficients of these encoders satisfy conditions, derived in the Appendix, which do not severely restrict the set of available filter transfer functions. The conditions do, however, preclude uniform filter encoders of orders 4, 8, 12, etc.

If t_0 is the encoder delay, the odd integer u_k is a scaled approximation to $y(k\tau - t_0)$. To determine the scaling factor, we observe that $x(k\tau)$, the ΔM approximation to $y(k\tau)$, is related to w_k by $x(k\tau) = \delta w_k$. Further, since the filter provides relatively distortionless gain over the signal bandwidth, it expands the amplitude scale of w_k by approximately the amount of the dc gain,

$$I = \sum_{i=0}^{N-1} g_i. \tag{2}$$

Thus, $(\delta/I)u_k$ is an approximation to $y(k\tau - t_0)$ and, with u_k ranging over odd integers, the signal levels represented by the input to the word generator are in the set

$$\cdots, \quad -3\frac{\delta}{I}, \quad -\frac{\delta}{I}, \quad \frac{\delta}{I}, \quad 3\frac{\delta}{I}, \cdots, \tag{3}$$

with quantizing step size $2\delta/I$. Because the scaling by δ/I is approximate, we admit an additional scale factor, γ, which brings the PCM representation optimally close to $y(t)$ in the mean square sense. The actual step size of the filter output is therefore

$$d_0 = \frac{2\delta}{I}\gamma. \tag{4}$$

In Section 7.5, we show that γ, whicb depends on g_i, is close to unity for encoders of practical interest.

Figure 3 shows the mapping of the filter output into M-bit code words. To eliminate α information bits from the binary representation of u_k, the word generator truncates the $\alpha + 1$ least significant bits. (With u_k odd at PCM sampling instants, the least significant bit always has value one and hence conveys no information.) In the absence of amplitude overload,

$$|u_k| \leqq 2^{M+\alpha} - 1$$

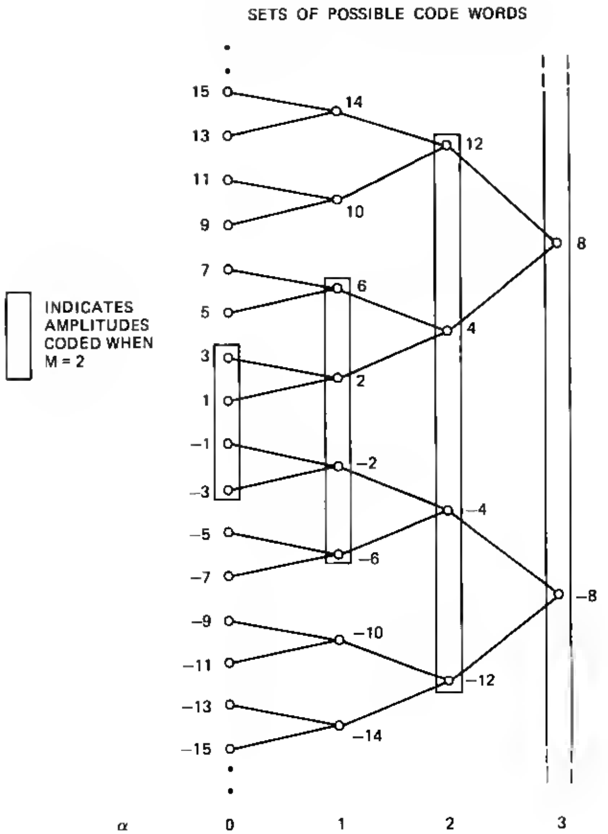

Fig. 3—Word generator roundoff procedure.

and the PCM code word consists of the $M - 1$ least significant bits of the truncated binary representation† of u_k and the sign bit. When

$$|u_k| > 2^{M+\alpha} - 1,$$

the transmitted code word is either the most positive or most negative M-bit word. A decoder recovers the integer code words of Fig. 3 by appending a one and α zeros to the least significant end of the PCM word.

With the truncation of each information bit, the step size increases by a factor 2 so that, with α information bits truncated, the PCM code

† In Section V, we point out an advantage of the twos complement binary format.

words represent signal levels in the set

$$\pm 2^\alpha \frac{d_o}{2}, \pm 3 \cdot 2^\alpha \frac{d_o}{2}, \pm 5 \cdot 2^\alpha \frac{d_o}{2}, \cdots, \pm (2^M - 1) \cdot 2^\alpha \frac{d_o}{2} \tag{5}$$

with step size

$$d = 2^\alpha d_o = 2^\alpha \frac{2\delta}{I} \gamma \tag{6}$$

and maximum amplitude $(2^M - 1)2^\alpha \delta\gamma/I$.

IV. UNIFORM DIGITAL FILTERS

The value of I, the dc gain of the digital filter, is crucial in determining the character of the overall PCM quantizing noise. With the filter coefficients all integers, I may be regarded as a measure of coefficient quantization. A large value of I corresponds to fine quantization because it allows considerable freedom in choosing g_i. To obtain a filter transfer function that approximates with arbitrary accuracy the optimum transfer function with respect to granular noise,[1] an arbitrarily high value of I is required. On the other hand, amplitude overload noise increases rapidly with I because the dynamic range of the encoder is nearly proportional to I^{-1}.

The rapid increase in amplitude overload noise as a function of I leads us to focus our attention on the uniform filter,

$$g_i = 1; \quad i = 0, 1, \cdots, N - 1, \tag{7}$$

for which $I = N$, resulting in the greatest dynamic range attainable with an Nth order filter with all coefficients of the same polarity. (We exclude from consideration filters with $g_i = 0$ for one or more i.) Reference 1 suggests that, for high sampling rates, the coefficients of the optimum filter with respect to granular noise are nearly equal and that the difference in granular noise rejection between this optimum filter and the uniform filter is marginal. This observation suggests that encoders with uniform filters, because they minimize amplitude overload noise and produce near minimal granular noise, are nearly optimal with respect to total quantizing noise. Further support for this speculation is given later.

In the frequency domain, the uniform filter transfer function is $\sin(\pi N f/f_s)/\sin(\pi f/f_s)$ and the filter rejects increasing amounts of ΔM granular noise as N increases. So long as f_s/N is large relative to $2W$, the signal component of $x(t)$ is undistorted by the filter; but, as f_s/N approaches $2W$, distortion of in-band signal components becomes

significant, and overall performance deteriorates with increasing N. Thus, if the advantages of very high-order filtering are sought, designs more sophisticated than eq. (7) are required.

V. IMPLEMENTATION

Besides possessing noise-rejection properties, uniform filters admit considerable hardware economies relative to other designs. With all coefficients unity, no multiplication is required, and each filter output is merely the sum of N successive counter levels. Therefore, one may implement the uniform filter as a resettable accumulator, thereby eliminating the delay line of Fig. 1, as well as the multipliers. To obtain a PCM sample, the coder sets the accumulator to the current level of the up/down counter and adds to the accumulator the next $N - 1$ counter levels.

Because the addition of N numbers is required only once for each PCM sample, and because $f_s/2W$, the number of ΔM samples per PCM sample, is generally much greater than N, it is possible to time-share a single accumulator among many signal channels. With inputs presented to the accumulator at the ΔM rate, the number of channels sharing a single accumulator may be as high as $f_s/2WN$. Hence, in terms of hardware requirements, the filter order, N, determines time-sharing capacity rather than the number of circuits necessary to realize a single encoder.

In addition to adding counter levels into an accumulator and truncating least significant bits of the sum, the encoder must detect amplitude overload and generate the most positive or most negative code word when the word generator is overloaded.

It must also restore the proportionality of the counter level, w_k, to the ΔM approximation, $x(k\tau)$, after each instance of counter overload. The wrap-around property of twos complement arithmetic ensures this proportionality whenever $|x(k\tau)| < (2^{M+\alpha} - 1)\delta$. On the other hand, a saturating counter would require special measures to restore tracking after each instance of counter overload.

VI. ENCODER PERFORMANCE

6.1 *Figure of Merit and Design Specifications*

An ideal decoder of the encoder output sequence obtains $\hat{y}(t)$ (defined in Section 7.1), a delayed, noisy approximation to the analog input $y(t)$. We define the quantizing noise power of $\hat{y}(t)$ to be

$$N_T = E\{[\gamma\hat{y}(t) - y(t - t_0)]^2\} \tag{8}$$

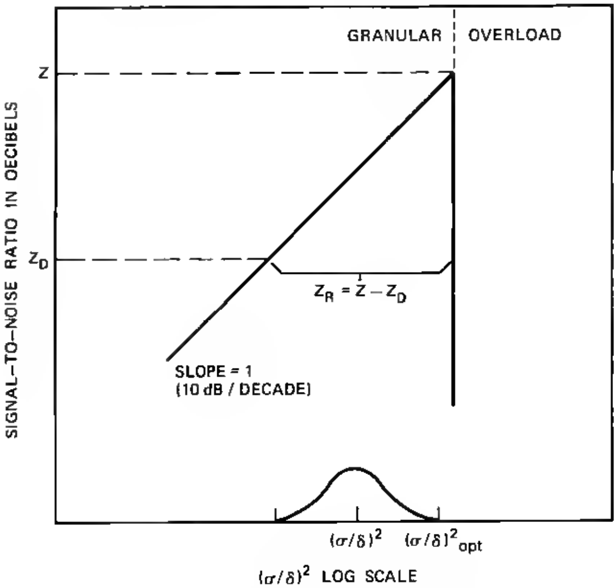

Fig. 4—Relationship of design objectives (Z_D,Z_R) to figure of merit (Z).

where E denotes expectation, t_0 is the encoder delay, and γ is the mean-square optimum scaling factor. For each digital filter, PCM word length, and ΔM sampling rate there is a unique combination of values of δ, the ΔM step size, and α, the word generator parameter, that results in minimal N_T. We choose as a figure of encoder merit the ratio of signal power to this minimum noise power,†

$$Z = \frac{E\{[y(t - t_0)]^2\}}{\min\limits_{\alpha,\delta} N_T}. \tag{9}$$

In the design of a practical encoder, typical specifications include a signal-to-noise ratio design goal, Z_D, and a range, Z_R, of input powers over which the actual signal-to-noise ratio must equal or exceed Z_D. The practical significance of our figure of merit is found in the approximation

$$Z \approx Z_D + Z_R \tag{10}$$

† N_T is a convex function of δ and α. In our numerical work we have used simple search techniques to find $\min_{\alpha,\delta} N_T$.

where each quantity is measured in decibels. For example, an encoder for which $Z = 55$ dB will actually attain this signal-to-noise ratio for a single level of input power, and will maintain a signal-to-noise ratio of 35 dB or better over a range of 20 dB in signal power.

Equation (10) is derived from Fig. 4, an approximation to the dependence of signal-to-noise ratio on input level. If (σ/δ) exceeds $(\sigma/\delta)_{\text{opt}}$, the optimum ratio of rms input to ΔM step size, overload noise predominates in the distortion and the signal-to-noise ratio falls rapidly as σ^2 increases. On the other hand, with $(\sigma/\delta) < (\sigma/\delta)_{\text{opt}}$, granular noise predominates and, with δ fixed, is essentially independent of σ. Hence, the signal-to-noise ratio is proportional to σ^2 in the granular region.

If the ensemble of input power levels is log-normally distributed, as in models used for speech signals,[6] $10 \log (\sigma/\delta)^2$ is a normal random variable, the mean value of which we denote by $10 \log (\bar{\sigma}/\delta)$. Hence, the probability that the signal-to-noise ratio exceeds Z_D is maximum when δ_D, the design value of the step size, is chosen such that

$$10 \log (\bar{\sigma}/\delta_D)^2 = 10 \log (\sigma/\delta)^2_{\text{opt}} - \tfrac{1}{2} Z_R. \tag{11}$$

That is, $10 \log (\bar{\sigma}/\delta_D)^2$ is the midpoint of the design range of length Z_R.

6.2 *Performance Characteristics*

Figure 5 shows a typical set of performance curves, computed according to the theory presented in Section VII. The curves pertain to 11-bit encoding of Gaussian signals having a truncated first-order Butterworth power spectrum, where the ratio of 3-dB frequency to cutoff frequency is 0.25. This type of process has been used to model band-limited speech.[7] The performance curves show the figure of merit, Z, of uniform filter encoders of various orders as a function of $f_s/2W$, the ΔM sampling rate expressed as a multiple of the PCM rate.

The choice of a specific encoder configuration represents a compromise between the advantages of low ΔM speed and low filter order. The nature of this compromise is illustrated in Fig. 6, which shows combinations of ΔM speed and filter order that satisfy two quality objectives. The broken curves relate ΔM speed to the maximum filter order consistent with sharing the accumulator described in Section V among 24, 48, and 96 signal channels, respectively. All design points to the right of a broken line are permissible for the given number of multiplexed channels.

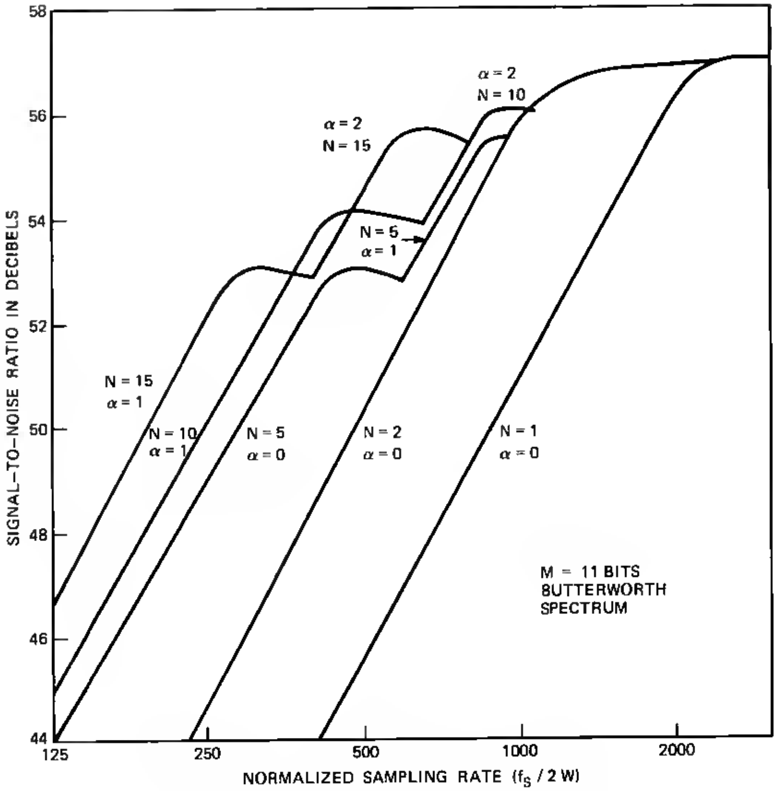

Fig. 5—Performance characteristics, $M = 11$ bits, Butterworth spectrum.

6.3 *Dependence of Performance on Design Parameters*

Figure 5 demonstrates two types of variation of Z with f_s: Z rising with a slope of 20 dB/decade, and Z flat or decreasing slowly with f_s. The first type of behavior occurs when amplitude overload is negligible and slope overload controls the optimum ΔM step size. In this case, the optimum step size varies approximately as $1/f_s$, and the decrease continues until amplitude overload becomes significant. When amplitude overload is the predominant overload noise, the optimum step size is essentially constant and the slightly negative slope of Z indicates that an increase in f_s results in an increase in the granular noise correlation from sample to sample, leading to a greater proportion of the ΔM granular noise power in the passband of the filter.

The flat portions of the curves represent transition regions to higher

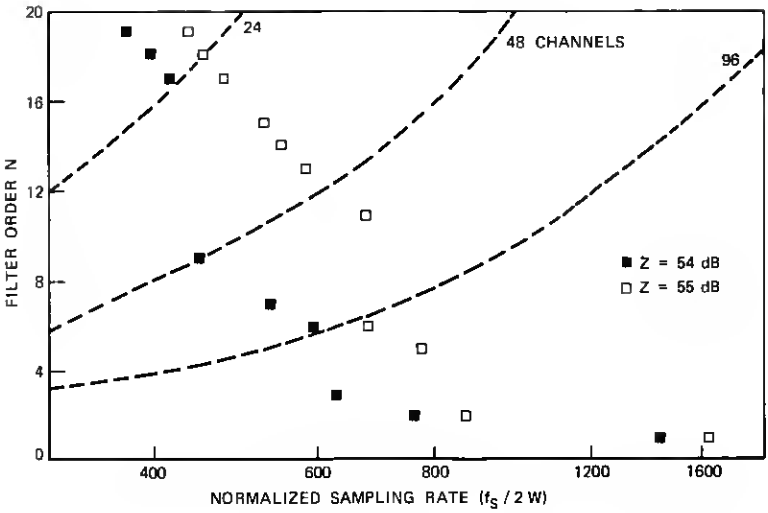

Fig. 6—Design alternatives derived from Fig. 5, $M = 11$ bits, Butterworth spectrum.

values of α, at which the increased roundoff noise of the word generator is offset by improved immunity to amplitude overload. As f_s increases indefinitely, Z approaches the maximum signal-to-noise ratio associated with uniform PCM encoding of Gaussian signals.

In Figs. 7 and 8, we see that the shapes of the characteristic curves are essentially invariant with the numher of bits in the PCM code. In Fig. 7, which pertains to 11-bit encoding of signals with a flat spectrum, amplitude overload effects occur at points that are approximately $10 \log (2^{11}/2^8) = 18$ dB higher in signal-to-noise ratio and further to the right by the factor $2^{11}/2^8$ in sampling rate, relative to the corresponding points in Fig. 8, which pertains to 8-bit encoding of the same input process.

Figures 5, 7, and 8 also demonstrate the effect of filter order. Wben f_s is quite low and amplitude overload is negligihle, Z increases monotonically with N. However, the value of f_s at which amplitude overload hecomes significant decreases as N increases, and the earlier transitions from $\alpha = 0$ to $\alpha = 1$, $\alpha = 1$ to $\alpha = 2$, etc., lead to the crossovers.

Figures 5 and 7 relate to the same PCM word length but different signal spectra. The principal difference hetween the two sets of curves is a scale change of the horizontal axis. In Fig. 5, the axis is shifted to the left relative to Fig. 7 hy the factor 1.6, which is the ratio by which

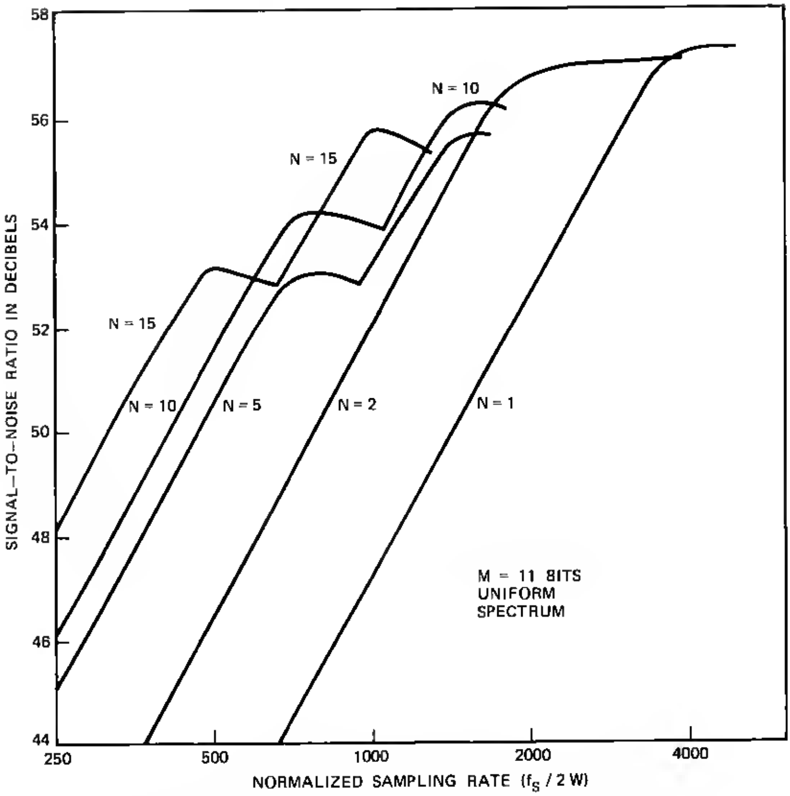

Fig. 7—Performance characteristics, $M = 11$ bits, uniform spectrum.

the rms slope of signals having the uniform spectrum exceeds the rms slope of signals having the Butterworth spectrum.

VII. QUANTIZING NOISE ANALYSIS

7.1 *Noise Components*

To reconstruct an analog signal from the sequence of word generator outputs, we first recover one of the integers shown in the αth column of Fig. 3 by appending a one and α zeros to the least significant end of each code word. We next multiply the sequence of integers by the nominal scale factor δ/I and denote the resulting sequence by $\{\hat{y}_j\}$. Finally, we perform ideal interpolation of $\{\hat{y}_j\}$ to obtain the continuous waveform

$$\hat{y}(t) = \sum_{j=-\infty}^{\infty} \hat{y}_j \frac{\sin 2\pi W(t - j/2W)}{2\pi W(t - j/2W)}. \tag{12}$$

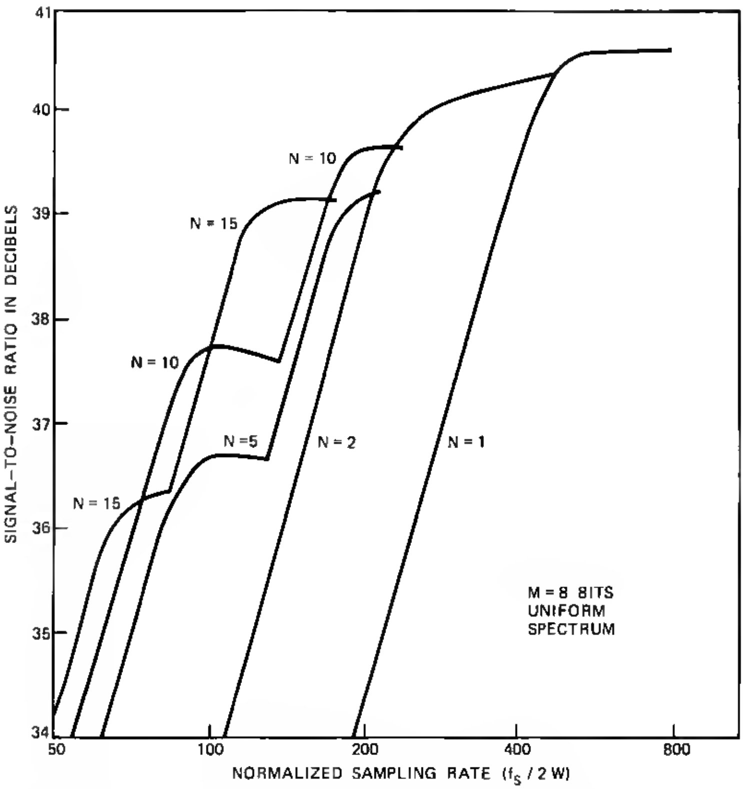

Fig. 8—Performance characteristics, $M = 8$ bits, uniform spectrum.

Our purpose, in this section, is to investigate the difference between $\hat{y}(t)$ and the encoder input, $y(t)$, when this input is a sample function of a zero-mean stationary Gaussian random process. Our measure of distortion is the total quantizing noise power, N_T, defined in eq. (8). Because $y(t)$ is stationary, N_T is independent of time, and in the sequel we omit time arguments and subscripts from the notation of signals when there is no risk of ambiguity. Although γ in eq. (8) is a complicated function of signal statistics and encoder design parameters, the introduction of the preliminary scaling factor, δ/I, leads to $\gamma \approx 1$ in situations of greatest interest. (See Section 7.5.) The other constant in eq. (8) is t_0, and we observe that if the filter coefficients have even symmetry ($g_i = g_{N-1-i}$), the filter delay is

$$t_0 = (N - 1)\tau/2, \tag{13}$$

one-half the filter memory span.

To derive N_T as a function of the encoder design parameters, we recognize $\hat{y}$ as the sum of a signal term and four noise terms. We begin by writing

$$\hat{y} = \frac{\delta}{I}(u + n_w) \tag{14}$$

where u is the filter output and n_w represents the difference between the input and output of the word generator. In studying $(\delta/I)u$, the filtered ΔM signal, it is customary to identify granular and slope overload components of the ΔM quantizing noise, in the manner indicated in Fig. 2. If we rewrite eq. (1) as $u = g * (x/\delta)$, with $*$ denoting convolution and x denoting the approximation signal in the delta modulator feedback loop, we obtain

$$\frac{\delta}{I}u = \frac{g}{I} * x = \frac{g}{I} * (y + n_G + n_S) \tag{15}$$

where n_G and n_S are ΔM granular and slope overload noise, respectively.

The distortion introduced by the word generator, $(\delta/I)n_w$, may itself be resolved into two components, namely, n_A, which accounts for amplitude overload, and n_R, which represents tbe roundoff effect. These observations lead us to a representation of the quantizing noise signal as the sum of four noise components:

$$(\gamma\hat{y} - y) = \left[\frac{\gamma g}{I} * (y + n_G) - y\right] + \frac{\gamma g}{I} * n_S + \gamma n_A + \gamma n_R. \tag{16}$$

The term in square brackets is the filtered granular noise, and tbe remaining terms are slope overload noise, amplitude overload noise, and word generator roundoff noise, respectively. In tbis paper we evaluate the expected square of eq. (16) by assuming that the expected product of each pair of terms is negligible relative to the sum of the two mean square values. Thus we express the total quantizing noise as tbe sum of four noise powers

$$N_T = N_G + \gamma^2 N_S + \gamma^2 N_A + \gamma^2 N_R \tag{17}$$

in which each term is the expected square of a term in eq. (16).

When the total quantizing noise is low, we are justified in approximating the average cross products of eq. (16) by zero because: (*i*) granular noise and roundoff noise are zero during overload bursts; (*ii*) each type of overload occurs with low probability and the probability of their joint occurrence is negligible; and (*iii*) we bave found that $|E(n_G n_R)|$ is many orders of magnitude lower than $N_G + N_R$ when $M \geqq 4$.

7.2 *ΔM Granular Noise and Slope Overload Noise*

Expanding the square of the term in brackets in eq. (16), we obtain

$$N_G = E(y^2) - 2\frac{\gamma}{I}\sum_{i=0}^{N-1} g_i R_{xy}(i\tau - t_0) + \left(\frac{\gamma}{I}\right)^2 \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} g_i g_j R_{xx}(i\tau - j\tau) \quad (18)$$

in which the ΔM correlation functions,

$$R_{xy}(\tau) = E[y(t)x(t + \tau)], \qquad R_{xx}(\tau) = E[x(t)x(t + \tau)]$$

are derived in Ref. 8, under the assumption that overload never occurs. To use the results of Ref. 8 and also account for overload, we should multiply N_G in eq. (18) by the probability that overload is absent. For the applications that interest us, the probability is greater than 0.99, and we adopt eq. (18) as an approximation to N_G that overestimates the granular noise component of N_T by no more than one percent. In Section 7.4, we similarly overestimate N_R.

To compute N_S, we adopt the assumption of previous authors[7,9] that essentially all of the ΔM slope overload noise power is in the signal band of $y(t)$ so that $N_S = E\{[(g/I) * n_S]^2\} \approx E(n_S^2)$. In our numerical analysis, we have followed Protonotarios,[9] who derives $E(n_S^2)$ as a function of

$$S = \frac{\delta}{\tau}\left\{E\left[\frac{dy}{dt}\right]^2\right\}^{-\frac{1}{2}}, \quad (19)$$

the ratio of the maximum slope of $x(t)$ to the rms slope of $y(t)$. For high values of S, N_S is proportional to $S^{-5}\exp[-\frac{1}{2}S^2]$.

7.3 *Amplitude Overload Noise*

The maximum output of the word generator is $2^\alpha(2^M - 1)$. Assuming there is no granular or slope overload noise during amplitude overload intervals,

$$\begin{aligned} n_A &= \frac{g}{I} * y - \frac{\delta}{I}2^\alpha(2^M - 1); && g * y > 2^\alpha(2^M - 1)\delta \\ &= \frac{g}{I} * y + \frac{\delta}{I}2^\alpha(2^M - 1); && g * y < -2^\alpha(2^M - 1)\delta \\ &= 0; && |g * y| \leqq 2^\alpha(2^M - 1)\delta. \quad (20)\end{aligned}$$

Because $(g/I) * y$ is a sample function of a zero-mean Gaussian process

with variance

$$\sigma_F^2 = \frac{1}{I^2} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} g_i g_j R_{yy}(i\tau - j\tau), \tag{21}$$

the mean square value of eq. (20) is

$$\begin{aligned} N_A &= (2/\pi)^{\frac{1}{2}} \int_A^\infty \sigma_F^2 (x - A)^2 \exp\left(-\tfrac{1}{2}x^2\right) dx \\ &= \sigma_F^2 \{(1 + A^2) \operatorname{erfc}(2^{-\frac{1}{2}}A) - (2/\pi)^{\frac{1}{2}} A \exp\left(-\tfrac{1}{2}A^2\right)\} \end{aligned} \tag{22}$$

in which A is the amplitude overload factor,

$$A = \frac{2^\alpha(2^M - 1)\delta}{I\sigma_F}. \tag{23}$$

7.4 *Word Generator Roundoff Noise*

With u the filter output and $z(\cdot)$ the mapping shown in Fig. 3,

$$N_R = \frac{\delta^2}{I^2} E[(z - u)^2]. \tag{24}$$

Because u is an odd integer, we have the binary number representation of $u > 0$,

$$u = \sum_{i=1}^{\infty} b_i 2^i + 1$$

where $b_i = 0$ or 1. The word generator truncates $b_\alpha b_{\alpha-1} \cdots b_1 1$ from this representation and z is obtained by replacing these digits with $10 \cdots 0 = 2^\alpha$.

Hence

$$z = \sum_{i=\alpha+1}^{\infty} b_i 2^i + 2^\alpha \tag{25}$$

and

$$\begin{aligned} z - u &= 0; & \alpha &= 0 \\ &= 2^\alpha - 1 - \sum_{i=1}^{\alpha} b_i 2^i; & \alpha &\geqq 1. \end{aligned} \tag{26}$$

For $u < 0$ the odd symmetry of Fig. 3 implies $z(u) = -z(-u)$. Hence $z - u$ is an odd integer in $[-(2^\alpha - 1), 2^\alpha - 1]$ and the ex-

pectation in eq. (24) is a weighted average of the integers $1^2, 3^2, \cdots, (2^\alpha - 1)^2$. From this observation, we immediately obtain the bounds

$$\frac{\delta^2}{I^2} \leqq N_R \leqq \frac{\delta^2}{I^2}(2^\alpha - 1)^2. \tag{27}$$

With $\alpha = 1$, the bounds are equal and $N_R = \delta^2/I^2$.

For $\alpha > 1$, we evaluate N_R only for coders with uniform digital filters. All odd integers are possible outputs of such filters. That is, $\Pr\{u = 2n + 1\} > 0$ for all n and, for low-noise encoding, this probability is quite nearly constant over a set of $2^{\alpha-1}$ consecutive integers. When scaled to the amplitude range of $y(t)$, such a set of filter outputs lies in an interval of length

$$\frac{2\delta}{I} 2^{\alpha-1}\gamma = d/2, \tag{28}$$

a small fraction of σ (typically of the order of $4\sigma/2^M$). For $M \geqq 5$, the envelope of $\Pr\{u = 2n + 1\}$ has approximately the Gaussian shape of the probability density of $y(t)$ and a piecewise constant approximation to this density over intervals of length d or less leads to highly accurate expressions for quantizing noise power.[10]

Over intervals of length $d/2$, $(z - u)^2$ takes on all the values 1^2, $3^2, \cdots (2^\alpha - 1)^2$ either in ascending or descending order. Hence, the piecewise constant approximation to $\Pr\{u = 2n + 1\}$ reduces the expectation in eq. (24) to an unweighted average of these $2^{\alpha-1}$ integers,

$$N_R = \frac{\delta^2}{I^2} 2^{-(\alpha-1)} \sum_{i=1}^{2^{\alpha-1}} (2i - 1)^2 = \frac{\delta^2}{I^2} \frac{(4^\alpha - 1)}{3}. \tag{29}$$

Noting that eq. (6) admits the expression

$$\gamma^2 \left(\frac{\delta}{I}\right)^2 = \frac{d^2}{4(4^\alpha)},$$

we summarize the results of this section as follows:

$$\begin{aligned}
\gamma^2 N_R &= 0; && \alpha = 0 \\
&= \frac{d^2}{16}; && \alpha = 1 \\
&= \frac{d^2}{12}(1 - 4^{-\alpha}); && \alpha \geqq 0, \quad \text{uniform filter encoders.}
\end{aligned} \tag{30}$$

The last line indicates that $N_R \to d^2/12$ as $\alpha \to \infty$. This limit is the granular noise associated with instantaneous PCM encoding of samples ranging over the continuum.

7.5 *The Optimum Scale Factor*

To complete our quantizing noise analysis and establish the validity of prescaling the word generator output by δ/I, we show that γ, the additional scaling factor that brings the amplitude of $\hat{y}(t)$ optimally close to that of $y(t)$ in the mean square sense, is nearly unity in designs of practical interest. Specifically, we derive the inequalities,

$$(1-b)\gamma_0 < \gamma < \gamma_0 \tag{31}$$

where b is of the order of magnitude of the noise-to-signal ratio and

$$\gamma_0 = \frac{\sum_{i=0}^{N-1} \sum_{j=0}^{N-1} g_i g_j R_{yy}(i\tau - t_0)}{\sum_{i=0}^{N-1} \sum_{j=0}^{N-1} g_i g_j R_{yy}(i\tau - j\tau)}. \tag{32}$$

Clearly as $f_s \to \infty$, all of the covariances in eq. (32) approach σ^2 and hence $\gamma_0 \to 1$. In all of the numerical examples considered in Section VI, the sampling rates are so high that γ_0 is quite nearly unity. For example, all of the points plotted in Fig. 6 correspond to $0.99 < \gamma_0 < 1.01$.

By definition, γ is the value of c that minimizes

$$E[(c\hat{y} - y)^2] = E\left[\left(\frac{cg}{I} * (y + n_G) - y + cn_0\right)^2\right]. \tag{33}$$

The expression in square brackets on the right side is identical to eq. (16) with c replacing γ and cn_0 replacing the last three terms. Because we have assumed the correlation of n_0 and $[(cg/I) * (y + n_G) - y]$ to be zero, we may rewrite

$$E[(c\hat{y} - y)^2] = E\left[\frac{cg}{I} * (y + n_G) - y\right]^2 + c^2E(n_0^2). \tag{34}$$

Differentiating this equation with respect to c, and equating to γ the value of c that causes the derivative to be zero, we obtain

$$\gamma = \frac{E\left[y \frac{g}{I} * (y + n_G)\right]}{E\left[\frac{g}{I} * (y + n_G)\right]^2 + E(n_0^2)}$$

$$= \frac{\frac{1}{I} \sum_{i=0}^{N-1} g_i R_{xy}(i\tau - t_0)}{\frac{1}{I^2} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} g_i g_j R_{xx}(i\tau - j\tau) + N_S + N_A + N_R}. \tag{35}$$

Reference 8 demonstrates that, for low-noise encoding, the approximations $R_{xy} \approx R_{yy}$ and $R_{xx} \approx R_{yy} + R_{ee}$ [where $R_{ee}(\cdot)$ is the autocorrelation function of the unfiltered granular quantizing noise] are extremely precise. Thus eq. (35) becomes

$$\gamma = \frac{\frac{1}{I} \sum_{i=0}^{N-1} g_i R_{yy}(i\tau - t_0)}{\frac{1}{I^2} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} g_i g_j R_{yy}(i\tau - j\tau) + N_G' + N_S + N_A + N_R} \tag{36}$$

where

$$N_G' = \frac{1}{I^2} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} g_i g_j R_{ee}(i\tau - j\tau)$$

and is of the order of magnitude of N_G.[1] We now recognize that the first term in the denominator of eq. (36) is σ_F^2 [eq. (21)], the power of the filtered signal component of the ΔM approximation. If we divide numerator and denominator by this term (and substitute $\sum_i g_i$ for I) we obtain

$$\gamma = \frac{\gamma_0}{1 + b} \tag{37}$$

where we have defined

$$b = (N_G' + N_A + N_S + N_R)/\sigma_F^2 \tag{38}$$

which is the order of magnitude of N_T/σ^2. Equation (31) follows immediately from eq. (37).

VIII. SUMMARY AND CONCLUSION

Section VII presents the analytical steps enabling us to compute the figure of merit, Z, of a ΔM/PCM encoder. The rationale for using

this figure of merit is presented in Section VI (Fig. 4) along with results for some special cases of interest (Figs. 5, 7, and 8). The ultimate utility of these results is that they enable the designer to determine tradeoffs between ΔM sampling speed and digital filter order for specified values of encoder quality (e.g., Fig. 6).

We should reiterate the conditions assumed for the encoder in deriving our results. First, we have assumed that the digital filter output is at odd parity at every PCM sampling instant. Aside from simplifying the roundoff noise analysis, this condition appears to correspond to the simplest possible implementation of the PCM word generator. The primary design constraint it imposes is the prohibition of digital filters of orders 4, 8, 12, etc.

Second, we have assumed that the digital filter has uniform coefficients. This condition makes a complete noise analysis relatively straightforward and also leads to a simple filter implementation. Furthermore, it corresponds to a robust design that appears to be near optimal in all cases of practical interest. Attempts to demonstrate the latter point quantitatively have foundered on the difficulty of assessing the roundoff noise power (N_R) when the filter coefficients are nonuniform. If we assume that, for any step size, d, the roundoff noise

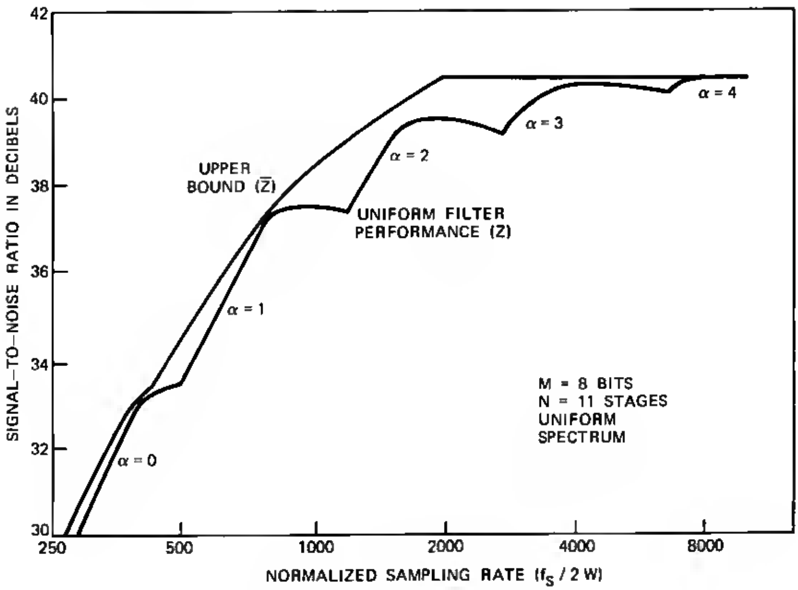

Fig. 9—Uniform filter assessment.

power is nondecreasing with the number of bits truncated, we can derive an upper bound on Z. A comparison of this upper bound, $\bar{Z}$, with the figure of merit resulting from uniform coefficients is shown in Fig. 9 for one case. Comparable results can be expected for other combinations of signal spectrum, PCM word length, and filter order.

APPENDIX

Odd-Parity Filter Outputs

With the encoder delay equal to one-half the filter memory span [eq. (13)], we consider symmetric coefficient sets

$$\begin{aligned} g_i &= g_{N-1-i}, \quad i = 0, 1, \cdots, \tfrac{1}{2} N - 1; \quad N \text{ even} \\ g_i &= g_{N-1-i}, \quad i = 0, 1, \cdots, \tfrac{1}{2}(N - 3); \quad N \text{ odd}. \end{aligned} \tag{39}$$

Such coefficients give equal weight to counter levels that are equally advanced or retarded with respect to $y(k\tau - t_0)$, the input value estimated at $t = k\tau$. Equation (39), when combined with eq. (1), implies

$$u_k = \sum_{i=0}^{\frac{1}{2}N-1} g_i[w_{k-i} + w_{k-(N-1)+i}]; \qquad N \text{ even} \tag{40}$$

$$u_k = g_{\frac{1}{2}(N-1)} w_{k-\frac{1}{2}(N-1)} + \sum_{i=0}^{\frac{1}{2}(N-3)} g_i[w_{k-i} + w_{k-(N-1)+i}]; \quad N \text{ odd}. \tag{41}$$

With the counter levels, w_i, alternating in parity, the two counter levels in square brackets in eq. (40) are of opposite parity because the difference in subscripts, $N - 1 - 2i$, is an odd number. Hence, their sum is odd. On the other hand, the two corresponding counter levels in eq. (41) have the same parity, and thus an even sum, because $N - 1 - 2i$ is even with N odd. These observations lead to the following necessary and sufficient conditions for u_k ranging over the set of odd integers:

Condition A: With N even, u_k is odd if and only if there is an odd number of odd coefficients in the set $g_0, g_1, \cdots, g_{\frac{1}{2}N-1}$.

Condition B: With N odd, u_k is odd at a PCM sampling instant if and only if $g_{\frac{1}{2}(N-1)}$ is odd and the low-speed clock is synchronized so that PCM sampling instants occur when $w_{k-\frac{1}{2}(N-1)}$ is odd. This synchronization can be achieved if the ratio of ΔM sampling rate to PCM sampling rate is an even integer.

In uniform filter encoders, Condition A is always satisfied when $N = 2, 6, 10$, etc. It can never be satisfied with $N = 4, 8, 12$, etc. For Condition B to be satisfied, the encoder must be synchronized such that w_{k-N+1} (the first term entering the accumulator described in Section V) is odd when $N = 1, 5, 9$, etc; w_{k-N+1} must be even when $N = 3, 7, 11$, etc.

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